Worcester County Mathematics League

Varsity Meet 4
February 26, 2014

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 4 – February 26, 2014
Round 1: Elementary Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

Note: a subscript indicates that number's base.
Bases are assumed to be positive integers ≥ 2.

1. How many prime numbers are there between $\sqrt{40}$ and $\sqrt{400}$?

2. The number 90^9 has 1900 different positive integral divisors. How many of these divisors are also perfect squares?

3. Change $\frac{13_{10}}{16_{10}}$ into a repeating decimal base 5. Indicate repeating digits with a bar; e.g. $0.1\overline{23} = 0.1232323\ldots$.
(The values of each place in base b are $\ldots b^2 b^1 b^0 . b^{-1} b^{-2} b^{-3} \ldots$)

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________

(3 pts.) 3. ________________ $_5$ [base 5 answer]

Varsity Meet 4 – February 26, 2014
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $x + 3y = 7$ and $5x - 7z = 15$, find the value of $15y + 7z$.

2. Let a, b, c be the three roots of the cubic polynomial equation

$$x^3 - 3x^2 - 16x + 6 = 0.$$

Compute $a^2 + b^2 + c^2$.

3. Paul and John went out for a bike ride and were 16 miles from home when Paul ran into a tree, damaging his bicycle beyond repair. They decide to return home with Paul starting on foot and John on his bicycle. After some time, John will leave his bicycle beside the road and continue on foot so that Paul can ride it the rest of the distance. Paul walks at 4 miles per hour and cycles at 10 miles per hour; John walks at 5 miles per hour and cycles at 12 miles per hour. For how many ***minutes*** should John ride the bicycle if they are both to arrive home at the same time?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________ minutes

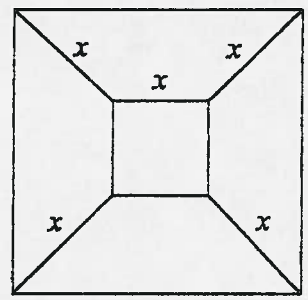
Varsity Meet 4 – February 26, 2014
Round 3: Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The side of a square is 4 cm longer than the side of an equilateral triangle. The perimeter of the square is 24 cm more than the perimeter of the triangle. Find the side length of the triangle, in cm.

2. In the figure, the area between the two squares can be written as Ax^2. Compute the value of A.



3. Suppose two spheres of radius 12 are externally tangent. A set of three spheres of radius $R > 0$ are mutually externally tangent, and each is also externally tangent to both spheres of radius 12. Compute R.

ANSWERS

(1 pt.) 1. ________________ cm

(2 pts.) 2. ________________

(3 pts.) 3. $R =$ ________________ units

Varsity Meet 4 – February 26, 2014
Round 4: Logarithms, Exponents, and Radicals

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify: $2 \cdot \left(5^{\sqrt{3}}\right) \cdot \left(5^{-\sqrt{3}}\right)$

2. Evaluate the sum

$$\log_2 8 + \log_4 8 + \log_8 8 + \log_{16} 8.$$

3. If $A = \log_{10} 75$ and $B = \log_{10} 135$, express $\log_{10} 24$ in terms of A and B in simplest form.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 4 – February 26, 2014
Round 5: Trigonometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify to a single trigonometric function:

$$\sec\theta - \sin\theta\tan\theta$$

2. Find the smallest positive angle x (in degrees) that satisfies the equation

$$\sin 20^\circ + \sin(x-20)^\circ = \sin(x+20)^\circ$$

3. Find all values of x (in radians), $0 \le x < 2\pi$, such that

$$\cos 2x = \left(\tan\frac{x}{2}\right)(1+\cos x).$$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. $x =$ __________________ ° [answer in degrees]

(3 pts.) 3. $x =$ __________________ [answer in radians]

5

Varsity Meet 4 – February 26, 2014
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. Find all values of x that satisfy the determinant equation

$$\begin{vmatrix} 3 & x & -1 \\ -x & x & -x \\ 1 & x & 3 \end{vmatrix} = 44.$$

2. Suppose that in base b, the numeral 143_b represents an integer which is the product of exactly two distinct primes. For how many positive integer bases $b < 40$ is this true?

3. Each face of a cube is painted either black or white. How many distinguishable ways are there to paint the cube? Paintings equivalent by rotation of the cube are indistinguishable.

4. Together Anne, Frank, and Kitty have \$31. Find the amount Kitty has if Frank has \$3 less than Anne, and Kitty has \$1 more than Anne and Frank combined.

5. Let $\triangle ABC$ be isosceles with $AB = AC$. Points D, E, and F are chosen on $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively such that $\triangle DEF$ is equilateral. If $m\angle AFE = 82°$ and $m\angle CED = 86°$, find $m\angle BDF$, in degrees.

6. If the sum $1! + 3! + 5! + \ldots + 99!$ were divided by 45, what remainder would be obtained?

7. If $2^y = 4 \cdot 2^x$ and $\dfrac{3^x}{27} = 9^y$, find the ordered pair (x, y).

8. Let $\triangle ABC$ be a right triangle with $AB = 4$, $BC = 5$, and $AC = 3$. A line from A is drawn through side $\overline{BC}$ to a point X outside $\triangle ABC$ such that $m\angle CAX = m\angle CBX$ and the intersection of $\overline{AX}$ and $\overline{BC}$ is between B and C. If $CX = 1.4$, find AX.

9. Find all possible values of x, $0 \le x < 360°$, that satisfy the following:

$$\begin{cases} \sin^2 x + \cos^2 y &= 1 \\ \sin^2 x - \cos^2 y &= 0 \\ \sin x + \cos x &< 0 \end{cases}$$

Varsity Meet 4 – February 26, 2014
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. ____________________

2. ____________________

3. ____________________

4. $ ____________________

5. ____________________ °

6. ____________________

7. (,)

8. ____________________

9. ____________________ °

Varsity Meet 4 – February 26, 2014
ANSWERS

ROUND 1
(Westborough, Tahanto, Quaboag)

1. 5
2. 250
3. $0.\overline{4012}_5$

ROUND 2
(Auburn, QSC, Algonquin)

1. 20
2. 41
3. 30 minutes

ROUND 3
(Westborough, Auburn, QSC)

1. 8
2. $2 + 2\sqrt{2}$ or equivalent
3. 72

ROUND 4
(Clinton, Burncoat, QSC)

1. 2
2. $25/4 = 6\frac{1}{4} = 6.25$
3. $3 - 2A + B$ or equivalent

ROUND 5
(Bartlett, Hudson, Hudson)

1. $\cos\theta$
2. $60°$
3. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ [need all 3, any order]

TEAM ROUND
(Algonquin, Hudson, Mass Acad, Burncoat, Algonquin, Leicester, Notre Dame Acad, QSC, Auburn)

1. $-\frac{11}{3}, 2$ [need both, either order]
2. 3
3. 10
4. \$16 or \$16.00
5. $84°$
6. 37
7. $(-7, -5)$
8. 4
9. $225°$

Varsity Meet 4 – February 26, 2014
FULL SOLUTIONS

ROUND 1

1. Note that $6 < \sqrt{40} < 7$ and $\sqrt{400} = 20$. The primes between the two are 7, 11, 13, 17, and 19. There are $\boxed{5}$ of them.

2. Since $90 = 2 \cdot 3^2 \cdot 5$, we have $90^9 = 2^9 \cdot 3^{18} \cdot 5^9$. We can confirm that this number has $(9+1)(18+1)(9+1) = 1900$ divisors.

   The divisors that are also perfect squares will have in their prime factorizations all even exponents (remember, 0 is even). For 2, the exponent can be one of $\{0, 2, 4, 6, 8\}$. For 3, the exponent can be one of $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$. For 5, it is the same as for 2. The total number of possibilities is therefore $5 \cdot 10 \cdot 5 = \boxed{250}$.

3. Begin to convert 13/16 into base 5 and then notice when the process repeats. Since $13/16 < 1$, the units digit is zero.

   The first digit after the decimal point is $\left\lfloor \frac{13}{16} \cdot 5 \right\rfloor = \left\lfloor \frac{65}{16} \right\rfloor = 4$. The remainder is $\frac{65}{16} - 4 = \frac{1}{16}$.

   The second digit after the decimal point is $\left\lfloor \frac{1}{16} \cdot 5 \right\rfloor = \left\lfloor \frac{5}{16} \right\rfloor = 0$. The remainder is $\frac{5}{16} - 0 = \frac{5}{16}$.

   The third digit after the decimal point is $\left\lfloor \frac{5}{16} \cdot 5 \right\rfloor = \left\lfloor \frac{25}{16} \right\rfloor = 1$. The remainder is $\frac{25}{16} - 1 = \frac{9}{16}$.

   The fourth digit after the decimal point is $\left\lfloor \frac{9}{16} \cdot 5 \right\rfloor = \left\lfloor \frac{45}{16} \right\rfloor = 2$. The remainder is $\frac{45}{16} - 2 = \frac{13}{16}$.

   At this point, we are left with $\frac{13}{16}$, which is what we started with. Therefore, the digits 4012 will repeat, and the base 5 representation for $\frac{13_{10}}{16_{10}}$ is $\boxed{0.\overline{4012}_5}$.

ROUND 2

1. Multiply the first equation by 5 and then subtract the second equation:

$$\begin{aligned} 5x + 15y &= 35 \\ -5x + 7z &= -15 \\ \hline 15y - 7z &= \boxed{20} \end{aligned}$$

2. From VIETA'S FORMULAS, $a+b+c=3$ and $ab+ac+bc=-16$. Therefore, we have that $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)=9+32=\boxed{41}$.

   [The roots of this cubic (-3, $3\pm\sqrt{7}$) are all real, but the formula is still applicable if any or all of the roots are complex! The method using Vieta's formulas is vastly preferred over solving for the roots of the polynomial, especially as the degree of the polynomial increases. In fact, it can be proven that there exists no formula to solve a general polynomial equation with degree ≥ 5.]

3. Suppose John bikes for x miles. Then, he must walk for $16-x$ miles; additionally, Paul must bike for $16-x$ miles and walk x miles. Given their speeds walking and biking, equate the total time and solve for x:

$$\begin{aligned} \frac{x}{12}+\frac{16-x}{5} &= \frac{x}{4}+\frac{16-x}{10} \\ \frac{16-x}{10} &= \frac{x}{6} \\ 96-6x &= 10x \\ 6 &= x. \end{aligned}$$

   Since John's biking speed is 12 miles per hour, he must bike for $6/12 = 1/2$ hour, or $\boxed{30}$ minutes.

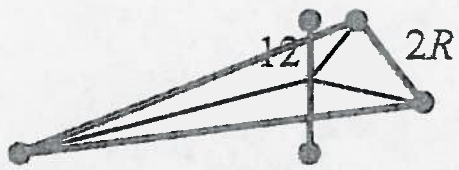
   [A possible pitfall is to equate John's biking and Paul's walking *times* instead of *distances*. The problem is that Paul does not immediately get the bike once John starts walking. This line of reasoning leads to the erroneous answer of 48 minutes.]

ROUND 3

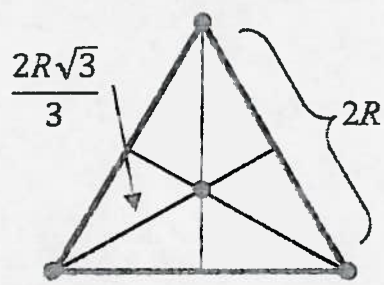
1. Let the side length of the triangle be x cm. Then, $3x+24=4(x+4)$, so $x=\boxed{8}$.

2. Utilize 45-45-90 right isosceles triangles (with side length ratio $1 : 1 : \sqrt{2}$) to find that the side length of the large square is $x(1+\sqrt{2})$. Therefore, the area of the large square is $x^2(3+2\sqrt{2})$. Subtract off the area of the small square (which is x^2) to find that the area between the squares is $x^2(2+2\sqrt{2})$. Therefore, $A = \boxed{2+2\sqrt{2}}$.

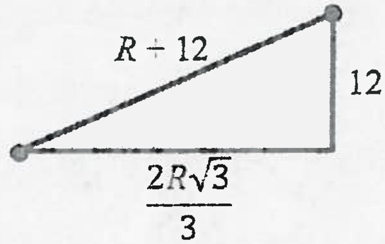
3. The centers of the spheres must be in the following configuration (dots indicate centers of spheres):



The two spheres of radius 12 are mutually tangent, so their centers are 24 units apart. The three spheres of radius R are mutually tangent, so their centers are at the vertices of an equilateral triangle of side length $2R$. A right triangle exists with one vertex as the center of a sphere of radius 12, another at the center of a sphere of radius R, and the third vertex at the center of the configuration. Next, consider the equilateral triangle described by the centers of the spheres of radius R:



Using 30-60-90 right triangles, the distance indicated in the above figure is $\frac{2R\sqrt{3}}{3}$. Now, solve the right triangle as described earlier:



One leg has length 12, the radius of one of the spheres in the pair. The length of the other leg was found in the second figure. The hypotenuse has length $R+12$ because we are given that the spheres of radius 12 and of radius R are externally tangent. Use the Pythagorean Theorem on this triangle to find that

$$\begin{aligned}\frac{4}{3}R^2+12^2 &= R^2+24R+12^2\\ \frac{1}{3}R^2 &= 24R\\ R &= 24\cdot 3\\ R &= \boxed{72}.\end{aligned}$$

ROUND 4

1. When multiplying with a common base, exponents add. Therefore, $2\cdot\left(5^{\sqrt{3}}\right)\cdot\left(5^{-\sqrt{3}}\right)=2\cdot 5^0=\boxed{2}$.

2. Use the change-of-base formula $\log_x y=\dfrac{\log y}{\log x}$ and the property that $\log(x^y)=y\log x$. Then,

$$\begin{aligned}\log_2 8+\log_4 8+\log_8 8+\log_{16} 8 &= \frac{3\log 2}{\log 2}+\frac{3\log 2}{2\log 2}+\frac{3\log 2}{3\log 2}+\frac{3\log 2}{4\log 2}\\ &= 3+\frac{3}{2}+1+\frac{3}{4}\\ &= \boxed{\frac{25}{4}}.\end{aligned}$$

3. Use the property that $\log(xy)=\log x+\log y$. Therefore, $A=\log 75=\log 3+2\log 5$ and $B=\log 135=3\log 3+\log 5$. Also, $\log 24=3\log 2+\log 3=\log 3+3-3\log 5$. Therefore, our answer is in the form $\log 24=3+x\cdot A+y\cdot B$. Equate the coefficients of the $(\log 3)$ and $(\log 5)$ terms to solve for x and y:

$$\begin{aligned}x+3y &= 1\\ 2x+y &= -3\end{aligned}$$

Solving gives $(x,y)=(-2,1)$, so $\log 24=\boxed{3-2A+B}$.

ROUND 5

1. Use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$:

$$\begin{aligned}
\sec\theta - \sin\theta\tan\theta &= \frac{1}{\cos\theta} - \sin\theta\cdot\frac{\sin\theta}{\cos\theta} \\
&= \frac{1-\sin^2\theta}{\cos\theta} \\
&= \frac{\cos^2\theta}{\cos\theta} \qquad \text{(Pythagoras)} \\
&= \boxed{\cos\theta}.
\end{aligned}$$

2. Use the addition formula for sine:

$$\begin{aligned}
\sin 20^\circ + \sin(x-20)^\circ &= \sin(x+20)^\circ \\
\sin 20^\circ + (\sin x^\circ\cos 20^\circ - \sin 20^\circ\cos x^\circ) &= \sin x^\circ\cos 20^\circ + \sin 20^\circ\cos x^\circ \\
\sin 20^\circ &= 2\sin 20^\circ\cos x^\circ \\
1/2 &= \cos x^\circ.
\end{aligned}$$

The smallest positive value of x satisfying $\cos x = 1/2$ is $\boxed{60^\circ}$.

3. **METHOD I:** The cosine addition formula $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$ suggests that $1+\cos x$ can be simplified by making the substitution $y = x/2$. Then, we have

$$\begin{aligned}
\cos 4y &= (\tan y)(1+\cos 2y) \\
\cos 4y &= \frac{\sin y}{\cos y}\cdot 2\cos^2 y \\
\cos 4y &= \sin 2y
\end{aligned}$$

Transform back to x and we have

$$\begin{aligned}
\cos 2x &= \sin x \\
1 - 2\sin^2 x &= \sin x \\
0 &= 2\sin^2 x + \sin x - 1 \\
0 &= (2\sin x - 1)(\sin x + 1)
\end{aligned}$$

Thus $\sin x$ is equal to $1/2$ or -1. The values of x between 0 and 2π that satisfy this are $\boxed{\pi/6, 5\pi/6, 3\pi/2}$.

METHOD II: Use the tangent half-angle formula

$$\tan(x/2) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

to get immediately that $\cos 2x = \sin x$. Then proceed as in Method I.

TEAM ROUND

1. The determinant of the matrix is $6x^2+10x$. Set this equal to 44:

$$\begin{aligned} 6x^2+10x-44 &= 0 \\ (3x+11)(2x-4) &= 0 \\ x &= \boxed{-11/3,\ 2} \end{aligned}$$

Plugging in, both values work.

2. Since $143_b = b^2+4b+3 = (b+1)(b+3)$, we are looking for twin primes; i.e. primes that differ by 2. Because of the presence of the 4 in 143_b, we know that $b > 4$. The relevant primes are $\ldots, 7, \mathbf{11}, \mathbf{13}, \mathbf{17}, \mathbf{19}, 23, \mathbf{29}, \mathbf{31}, 37, 41, \ldots$. There are $\boxed{3}$ sets of twin primes, corresponding to $b = 10, 16, 28$.

3. Divide into cases:

- 6 black, 0 white: 1 way.
- 5 black, 1 white: 1 way.
- 4 black, 2 white: 2 ways. The white faces can either be adjacent, or they can be opposite.
- 3 black, 3 white: 2 ways. The 3 white faces can either be opposite faces joined by any of the 4 equivalent remaining faces, or they can be three faces sharing a common vertex.
- The remaining 3 cases are symmetric to the first three cases, for 4 more ways.

In total, there are $1+1+2+2+2+1+1 = \boxed{10}$ ways to color the cube.

4. Let A be the amount of money Anne has, F be how much Frank has, and K be how much Kitty has. Then:

$$\begin{cases} A+F+K &= 31 \\ F+3 &= A \\ K &= A+F+1. \end{cases}$$

   METHOD I: Substituting, $(F+3)+F+([F+3]+F+1)=31$, so $F=6$. Therefore, $A=6+3=9$ and $K=9+6+1=\boxed{\$16}$.

   METHOD II: Alternatively, $K=31-(A+F)$ from the first equation, so combining with the third, $2K=31+1$ so $K=\boxed{\$16}$.

5. **METHOD I:** Straight angles sum to 180°, so $m\angle AEF=34°$. From $\triangle AFE$, find that $m\angle A=64°$. Since $\triangle ABC$ is isosceles, $m\angle B=m\angle C$ must be $\frac{1}{2}(180°-m\angle A)=58°$. Meanwhile, $m\angle DFB=38°$ by the straight angle. Since angles of a triangle sum to 180°, $m\angle BDF=180°-(38°+58°)=\boxed{84°}$.

   METHOD II: By the EXTERIOR ANGLE THEOREM, we have that

$$\begin{aligned} m\angle AFE+60° &= m\angle ABC+m\angle BDF \\ \text{and}\quad m\angle BDF+60° &= m\angle CED+m\angle ACB. \end{aligned}$$

   Since $\triangle ABC$ is isosceles with $AB=AC$, $m\angle ABC=m\angle ACB$. Subtract the above equations to find that

$$m\angle BDF-m\angle AFE=m\angle CED-m\angle BDF,$$

   so

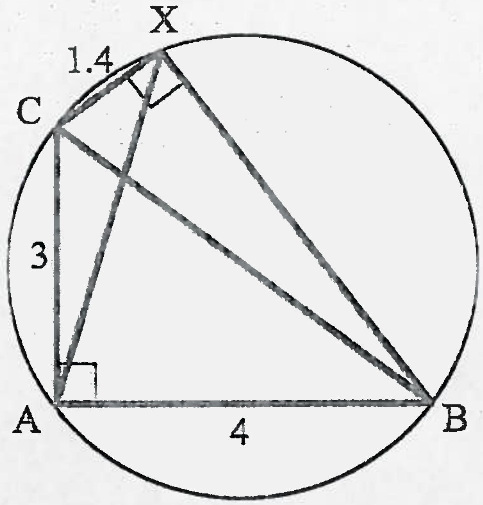
$$m\angle BDF=\frac{m\angle CED+m\angle AFE}{2}=\frac{86°+82°}{2}=\boxed{84°}.$$

6. Note that $45=3^2\cdot 5$. Since 7! contains factors of 3, 5, and 6, it is evenly divisible by 45. The same is true for all $N!$ with $N\geq 6$. Therefore, the only terms making a contribution to the remainder modulo 45 are 1!, 3!, and 5!.

   Since $1!+3!+5!=1+6+120=127$ and $127=2\cdot 45+37$, our desired remainder is $\boxed{37}$.

7. Express the terms with a common base: $2^y=2^{2+x}$ and $3^{x-3}=3^{2y}$. Therefore, $y=x+2$ and $x-3=2y$. Solve the system and $(x,y)=\boxed{(-7,-5)}$.

8. METHOD I: Consider the circumcircle of $\triangle ABC$. Since $\triangle ABC$ is a right triangle, the hypotenuse $\overline{BC}$ is a diameter of the circle.



Since we are given that $m\angle CAX = m\angle CBX$, X must be a point along arc BC [proof[1]]. Therefore, since $\overline{BC}$ is a diameter, $m\angle BXC = 90°$. By the Pythagorean Theorem, $BX = \sqrt{5^2 - (1.4)^2} = 4.8$.

Moreover, since quadrilateral $ABXC$ is cyclic, use PTOLEMY'S THEOREM to find the length AX.

$$\begin{aligned} (AC \cdot BX) + (AB \cdot CX) &= AX \cdot BC \qquad \text{(Ptolemy)} \\ (3 \cdot 4.8) + (4 \cdot 1.4) &= 5 \cdot AX \\ 20 &= 5 \cdot AX \\ \boxed{4} &= AX. \end{aligned}$$

METHOD II: Let the intersection of $\overline{AX}$ and $\overline{BC}$ be H. Using vertical angles at H, $\triangle AHC \sim \triangle BHX$ by AA. Now, by SAS similarity, $\triangle CHX \sim \triangle AHX$. Using equivalent angles from similar triangles, note that $m\angle BXC = m\angle BXA + m\angle AXC = m\angle ACH + m\angle ABC = 180° - m\angle CAB = 90°$. Since opposite angles ($\angle CAB$ and $\angle BXC$) of quadrilateral $ABXC$ sum to 180°, the quadrilateral is cyclic. Proceed with the Pythagorean Theorem and Ptolemy's Theorem as in Method I.

9. Adding the first two equations gives $2\sin^2 x = 1$, so $x = 45°, 135°, 225°$, or $315°$. Plugging each of these into the third statement shows that only $x = \boxed{225°}$ satisfies all three conditions.

[1]Clearly, all of the points along the arc satisfy the requirement that $m\angle CAX = m\angle CBX$ because both angles are one-half of the arc length CX. Now, we must show that there are no other points satisfying this condition. Consider any point X' along the arc BC. Since X' must be outside $\triangle ABC$, consider the ray of points along the line AX', starting from the intersection of $\overline{AX'}$ and $\overline{BC}$ and moving away from point A. As we move away from A, $m\angle CAX$ is invariant, but $m\angle CBX$ increases monotonically. Therefore, there can be only one point where $m\angle CAX = m\angle CBX$, and this is the point on the circumcircle of $\triangle ABC$.